Multiple positive and sign-changing solutions for nonlinear Schrodinger equations (Dynamics of spatio - temporal patterns for the system of reaction - diffusion equations)

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Multiple positive and sign-changing solutions for nonlinear Schrödinger equations

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0. Introduction

In this paper we consider the existence and multiplicity of solutions of the following nonlinear Schrödinger equations:

$$\begin{aligned} &-\Delta u + (\lambda^2 a(x) + 1)u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N, \\ &u(x) \in H^1(\mathbf{R}^N). \end{aligned} \tag{P_λ}$$

Here $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$, $p \in (1, \infty)$ if $N = 1, 2$ and $a(x) \in C(\mathbf{R}^N, \mathbf{R})$ is non-negative on $\mathbf{R}^N$. We consider multiplicity of solutions (including positive and sign-changing solutions) when the parameter λ is very large.

For $a(x)$, we assume

(a1) $a(x) \in C(\mathbf{R}^N, \mathbf{R})$, $a(x) \geq 0$ for all $x \in \mathbf{R}^N$ and the potential well $\Omega = int\, a^{-1}(0)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$ and $a^{-1}(0) = \overline{\Omega}$.

(a2) $0 < \liminf_{|x|\to\infty} a(x) \leq \sup_{x\in\mathbf{R}^N} a(x) < \infty$.

When λ is large, the potential well Ω plays important roles and the following Dirichlet problem appears as a limit of (P_λ):

$$\begin{aligned} &-\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega, \\ &u = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{0.1}$$

We remark that solutions of (P_λ) and (0.1) can be characterized as critical points of

$$\Psi_\lambda(u) = \int_{\mathbf{R}^N} \frac{1}{2}(|\nabla u|^2 + (\lambda^2 a(x) + 1)u^2) - \frac{1}{p+1}|u|^{p+1}\, dx : H^1(\mathbf{R}^N) \to \mathbf{R}, \tag{0.2}$$

$$\Psi_\Omega(u) = \int_\Omega \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1}\, dx : H_0^1(\Omega) \to \mathbf{R} \tag{0.3}$$

and it is known that (0.3) has an unbounded sequence of critical values (cf. ...)

Bartsch and Wang [**BW2**] and Bartsch, Pankov and Wang [**BPW**] studied such a situation firstly. Their assumptions on $a(x)$ and nonlinearity are more general and as a special case of their results we have

(i) There exists a least energy solution $u_\lambda(x)$ of (P_λ). Moreover $u_{\lambda_n}(x)$ converges strongly to a least energy solution of (0.3) after extracting a subsequence $\lambda_n \to \infty$ ([**BW2**]).

(ii) When $N \geq 3$ and $p \in (1, \frac{N+2}{N-2})$ is close to $\frac{N+2}{N-2}$, there exists at least $\mathrm{cat}(\Omega)$ positive solutions of (P_λ) for large λ ([**BW2**]). Here $\mathrm{cat}(\Omega)$ denotes Lusternik-Schnirelman category of Ω.

(iii) For any $n \in \mathbf{N}$, there exist n pairs of (possibly sign-changing) solutions $\pm u_{1,\lambda}(x), \cdots,$ $\pm u_{n,\lambda}(x)$ of (P_λ) for large $\lambda \geq \lambda(n)$. Moreover they converge to distinct solutions $\pm u_1(x), \cdots, \pm u_n(x)$ of (0.1) after extracting a subsequence $\lambda_n \to \infty$ ([**BPW**]).

Here we remark that in [**BW2**], [**BPW**] they consider mainly the case where Ω is connected.

In this paper we consider the case where Ω consists of 2 connected components:

$$\Omega = \Omega_1 \cup \Omega_2 \tag{0.4}$$

and we consider the multiplicity of positive and sign-changing solutions for large λ.

We have studied the multiplicity of positive solutions in our previous paper [**DT**], it is shown that there exist positive solutions $u_{1,\lambda}(x), u_{2,\lambda}(x), u_{3,\lambda}(x)$ of (P_λ) for large λ such that after extracting a subsequence $\lambda_n \to \infty$,

$$u_{1,\lambda_n}(x) \to \begin{cases} u_1(x) & \text{in } \Omega_1\ , \\ 0 & \text{in } \mathbf{R}^N \setminus \Omega_1, \end{cases} \qquad u_{2,\lambda_n}(x) \to \begin{cases} u_2(x) & \text{in } \Omega_2\ , \\ 0 & \text{in } \mathbf{R}^N \setminus \Omega_2, \end{cases}$$
$$u_{3,\lambda_n}(x) \to \begin{cases} u_1(x) & \text{in } \Omega_1\ , \\ u_2(x) & \text{in } \Omega_2, \\ 0 & \text{in } \mathbf{R}^N \setminus (\Omega_1 \cup \Omega_2), \end{cases}$$

strongly in $H^1(\mathbf{R}^N)$. Here $u_i(x)$ is a least energy solution of

$$\begin{aligned} -\Delta u + u &= u^p \quad \text{in } \Omega_i, \\ u &= 0 \qquad \text{in } \partial\Omega_i. \end{aligned} \tag{0.5}$$

In particular, (P_λ) has at least 3 positive solutions for large λ. See [**DT**] for the case Ω consists of multiple connected components: $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$.

We remark that a solution $u_i(x)$ of (0.5) is said to be a least energy solution if and only if

$$\Psi_{i,D}(u_i) = \inf\{\Psi_{i,D}(u);\ u(x) \in H_0^1(\Omega_i) \text{ is a non-trivial solution of (0.5)}\},$$

holds. Here $\Psi_{i,D}(u)$ is defined by

$$\Psi_{i,D}(u) = \int_{\Omega_i} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1}\,dx : H_0^1(\Omega_i) \to \mathbf{R}\,. \tag{0.6}$$

("D" stands for Dirichlet boundary conditions.) It is natural to ask the existence of a sequence of solutions of (P_λ) converging to solutions of (0.5) in each Ω_i, which may not be least energy solutions.

1. Results

First we deal with positive solutions. Our first theorem is the following

Theorem 1.1. *Assume (a1)–(a2),* (0.4) *and* $N \geq 3$. *Then there exists a* $p_1 \in (1, \frac{N+2}{N-2})$ *and* $\lambda_1 \geq 1$ *such that for* $p \in (p_1, \frac{N+2}{N-2})$ *and* $\lambda \geq \lambda_1$, (P_λ) *possesses at least* $\mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2) + \mathrm{cat}(\Omega_1 \times \Omega_2)$ *positive solutions.*

Remark 1.2. Since $\mathrm{cat}(\Omega_1 \cup \Omega_2) = \mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2)$, the argument of Bartsch-Wang [**BW2**] ensures $\mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2)$ positive solutions, which converges to a positive solution of (0.3) in one of components and to 0 elsewhere after extracting a subsequence $\lambda_n \to \infty$. We remark that our Theorem 1.1 ensures additional $\mathrm{cat}(\Omega_1 \times \Omega_2)$ positive solutions. We can also observe that these solutions converge to positive solutions in both components Ω_1, Ω_2.

Next we study the multiplicity of sign-changing solutions. When Ω consists of 2 components, we have two limit problems (0.5), which are corresponding to $\Psi_{i,D} : H_0^1(\Omega_i) \to \mathbf{R}$ $(i = 1, 2)$. It is well-known that each functional has an unbounded sequences of critical points $(u_j^{(i)}(x))_{j=1}^\infty \subset H_0^1(\Omega_i)$ $(i = 1, 2)$. A natural question is to ask for a given pair $(u_{j_1}^{(1)}(x), u_{j_2}^{(2)}(x))$ whether (P_λ) has a solution $u_\lambda(x) \in H^1(\mathbf{R}^N)$ converging to $u_{j_i}^{(i)}(x)$ in Ω_i and to 0 elsewhere. Here we try to give a partial answer to this problem. More precisely, we try to find a solution $u_\lambda(x) \in H^1(\mathbf{R}^N)$ which converges to $(u_1^{(1)}(x), u_j^{(2)}(x))$ after extracting a subsequence $\lambda_n \to \infty$. Here $u_1^{(1)}(x)$ is a mountain pass solution of (0.5) in Ω_1 and $u_j^{(2)}(x)$ is a minimax solution of (0.5) in Ω_2.

To find an unbounded sequence of critical values of a functional $I(u) \in C^1(E, \mathbf{R})$ defined on an infinite dimensional Hilbert space E, $\mathbf{Z}_2$-symmetry of $I(u)$ — $I(\pm u) = I(u)$ for all $u \in E$ — plays an important role. We remark that $\Psi_\lambda(u) \in C^1(H^1(\mathbf{R}^N), \mathbf{R})$ and a functional $\tilde{\Psi}(u_1, u_2) = \Psi_{1,D}(u_1) + \Psi_{2,D}(u_2) \in C^1(H_0^1(\Omega_1) \times H_0^1(\Omega_2), \mathbf{R})$, which is corresponding to (0.5) in Ω_1 and Ω_2, have different symmetries; $\Psi_\lambda(u)$ is $\mathbf{Z}_2$-symmetric and $\tilde{\Psi}(u_1, u_2)$ is $(\mathbf{Z}_2)^2$-symmetric, that is,

$$\Psi_\lambda(su) = \Psi_\lambda(u) \qquad \text{for all } s \in \mathbf{Z}_2 = \{-1, 1\},\ u \in H^1(\mathbf{R}^N),$$
$$\tilde{\Psi}(s_1 u_1, s_2 u_2) = \tilde{\Psi}(u_1, u_2) \text{ for all } s_1, s_2 \in \{-1, 1\},\ (u_1, u_2) \in H_0^1(\Omega_1) \times H_0^1(\Omega_2).$$

Note that $\mathbf{Z}_2$-action on $\Psi_\lambda(u)$ is corresponding to the following $\mathbf{Z}_2$-action on $\tilde{\Psi}(u_1, u_2)$

$$\tilde{\Psi}(su_1, su_2) = \tilde{\Psi}(u_1, u_2) \quad \text{for all } s \in \{-1, 1\},\ (u_1, u_2) \in H_0^1(\Omega_1) \times H_0^1(\Omega_2)$$

and there are no symmetries of $\Psi_\lambda(u)$ corresponding to the $\mathbf{Z}_2$-symmetry of $\tilde{\Psi}(u_1, u_2)$:

$$\tilde{\Psi}(u_1, \pm u_2) = \tilde{\Psi}(u_1, u_2). \tag{1.1}$$

We also remark that solutions $(u_1^{(1)}(x), u_j^{(2)}(x))$ are obtained using group action (1.1). Thus to construct solutions $u_\lambda(x)$ converging to $(u_1^{(1)}(x), u_j^{(2)}(x))$, we need to develop a kind of perturbation theory from symmetries and in this paper we use ideas from Ambrosetti [**A**], Bahri-Berestycki [**BB**], Struwe [**St**] and Rabinowitz [**R**] (See also Bahri-Lions [**BL**], Tanaka [**T**] and Bolle [**B**]). In [**A**, **BB**, **St**, **R**, **BL**, **T**], perturbation theories are developed for

$$-\Delta u = |u|^{p-1}u + f(x) \quad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain. They successfully showed the existence of unbounded sequence of solutions for all $f(x) \in L^2(\Omega)$ for a certain range of p.

Now we can give our second result.

Theorem 1.3. *Assume (a1)–(a2) and* (0.4). *Then $\Psi_{1,D}(u)$ and $\Psi_{2,D}(u)$ have critical values $c_{min}^{1,D}$ and $\{c_k^{2,D}\}_{k=1}^\infty$ with the following property: For any $k \in \mathbf{N}$ there exists $\lambda_2(k) \geq 1$ such that for any $\lambda \geq \lambda_2(k)$, (P_λ) has a solution $u_\lambda(x)$ such that*

(i) $\Psi_\lambda(u_\lambda) \to c_{min}^{1,D} + c_k^{2,D}$ as $\lambda \to \infty$.

(ii) For any given sequence $\lambda_\ell \to \infty$, we can extract a subsequence $\lambda_{n_\ell} \to \infty$ such that $u_{\lambda_{n_\ell}}$ converges to a function $u(x)$ strongly in $H^1(\mathbf{R}^N)$. Moreover $u(x)$ satisfies (0.5) *in $\Omega_1 \cup \Omega_2$, $u|_{\mathbf{R}^N \setminus (\Omega_1 \cup \Omega_2)} \equiv 0$ and $u(x) > 0$ in Ω_1.*

(iii) Moreover if the set of critical values of either $\Psi_{1,D}(u)$ or $\Psi_{2,D}(u)$ are discrete in a neighborhood of $c_{min}^{1,D}$ or $c_k^{2,D}$, then we have

$$\Psi_{1,D}(u|_{\Omega_1}) = c_{min}^{1,D}, \quad \Psi_{2,D}(u|_{\Omega_2}) = c_k^{2,D}.$$

Remark 1.4. It seems that discreteness of critical values of $\Psi_{i,D}(u)$ is not known; However we don't know any example that the set of critical values has interior points. We also remark that if the least energy solution of $\Psi_{1,D}(u)$ is non-degenerate — for example it holds for $\Omega = \{x \in \mathbf{R}^n;\, |x| < R\}$ $(R > 0)$ —, then critical values of $\Psi_{1,D}(u)$ are isolated in a neighborhood of $c_{min}^{1,D}$ and the assumption of (iii) holds.

When $N = 1$, we have a stronger result. We write $\Omega_1 = (a_1, b_1)$, $\Omega_2 = (a_2, b_2)$. For any j_1, $j_2 \in \mathbf{N}$ and $s_i \in \{-1, +1\}$ there exist unique solutions $u_i(x) = u_i(j_i, s_i; x)$ of (0.1) in Ω_i which possesses exactly j_i zeros in $\Omega_i = (a_i, b_i)$ and $s_i u_i'(a_i) > 0$. We have the following

Theorem 1.5. *Assume $N = 1$ and $\Omega_i = (a_i, b_i)$ $(i = 1, 2)$. Then for any j_1, $j_2 \in \mathbf{N}$ and $s_i \in \{-1, +1\}$ there exists a solution $u_\lambda(x)$ for large λ such that*

$$u_\lambda(x) \to u(x) \quad \textit{strongly in } H^1(\mathbf{R})$$

as $\lambda \to \infty$, where $u|_{\Omega_i}(x) = u_i(j_i, s_i; x)$ and $u|_{\mathbf{R}\setminus(\Omega_1\cup\Omega_2)}(x) = 0$.

In the following section, we give a variational formulation and give an idea of the proofs of Theorem 1.1. We refer [**ST**] for details of proofs of Theorems 1.1, 1.3 and 1.5.

2. Functional setting and variational formulation

(a) Reduction to a problem on an infinite dimensional torus

To find critical points of $\Psi_\lambda(u)$, we reduce our problem to a variational problem on an infinite dimensional torus. For $i = 1, 2$, we choose bounded open subset Ω_i' with smooth boundary such that

$$\Omega_i \subset\subset \Omega_i', \quad (i = 1, 2), \qquad \overline{\Omega_1'} \cap \overline{\Omega_2'} = \emptyset.$$

First we take local mountain pass approach due to del Pino and Felmer [**DF**] to find solutions concentrating only on $\Omega_1 \cup \Omega_2$. We choose a function $f(\xi) \in C^1(\mathbf{R}, \mathbf{R})$ such that for some $0 < \ell_1 < \ell_2$

$$\begin{aligned} &f(\xi) = |\xi|^{p-1}\xi \quad \text{for } |\xi| \le \ell_1, \\ &0 \le f'(\xi) \le \frac{2}{3} \quad \text{for all } \xi \in \mathbf{R}, \\ &f(\xi) = \frac{1}{2}\xi \quad \text{for } |\xi| \ge \ell_2. \end{aligned}$$

We set

$$g(x, \xi) = \begin{cases} |\xi|^{p-1}\xi & \text{if } \xi > 0 \text{ and } x \in \Omega_1' \cup \Omega_2', \\ f(\xi) & \text{if } \xi > 0 \text{ and } x \in \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'), \\ 0 & \text{if } \xi \le 0 \end{cases}$$

$$G(x, \xi) = \int_0^\xi g(x, s)\, ds.$$

In what follows we will try to find critical points of

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + (\lambda^2 a(x)+1)u^2\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx \\ &= \frac{1}{2}\|u\|^2_{\lambda,\mathbf{R}^N} - \int_{\mathbf{R}^N} G(x,u)\,dx.\end{aligned}$$

We can observe that $\Phi_\lambda(u) \in C^2(H^1(\mathbf{R}^N),\mathbf{R})$ satisfies $(PS)_c$ condition for all $c \in \mathbf{R}$. Moreover we have

Lemma 2.1. *Suppose that $(u_\lambda(x))_{\lambda\geq\lambda_0}$ is a family of critical points of $\Phi_\lambda(u)$ and assume that there exists constants m, $M > 0$ independent of λ such that*

$$m \leq \Phi_\lambda(u_\lambda) \leq M \quad \textit{for all } \lambda \geq 1.$$

Then we have

(i) $\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1} m \leq \|u_\lambda\|^2_{\lambda,\mathbf{R}^N} \leq \left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1} M$ *for all* $\lambda \geq 1$.

(ii) There exists $\lambda(M) \geq 1$ such that for $\lambda \geq \lambda(M)$, $u_\lambda(x)$ satisfies $0 \leq u_\lambda(x) \leq \ell_1$ for $x \in \mathbf{R}^N \setminus(\Omega_1' \cup \Omega_2')$. In particular, $g(x,u_\lambda(x)) = |u_\lambda(x)|^{p-1}u_\lambda(x)$ holds in $\mathbf{R}^N$ and $u_\lambda(x)$ is a solution of the original problem (P_λ).

(iii) After extracting a subsequence $\lambda_n \to \infty$, there exists $u \in H^1(\mathbf{R}^N)$ such that

$$\|u_{\lambda_n} - u\|_{\lambda_n,\mathbf{R}^N} \to 0 \quad \textit{as } n \to \infty.$$

Moreover $u(x)$ satisfies $u(x) \equiv 0$ in $\mathbf{R}^N \setminus(\Omega_1' \cup \Omega_2')$ and

$$-\Delta u + u = |u|^{p-1}u \quad \textit{in } \Omega_i, \tag{2.1}$$

$$u = 0 \qquad \textit{on } \partial\Omega_i \tag{2.2}$$

for $i = 1,2$. It also holds $\Phi_{\lambda_n}(u_{\lambda_n}) \to \Psi_{1,D}(u\big|_{\Omega_1}) + \Psi_{2,D}(u\big|_{\Omega_2})$ as $n \to \infty$.

Here and after we use notation

$$\|u_\lambda\|^2_{\lambda,O} = \int_O |\nabla u|^2 + (\lambda^2 a(x)+1)u^2\,dx$$

for an open set $O \subset \mathbf{R}^N$ and $\lambda > 0$.

Identifying $H^1(\Omega_1' \cup \Omega_2')$ and $H^1(\Omega_1') \oplus H^1(\Omega_2')$, we write $u = (u_1,u_2) \in H^1(\Omega_1' \cup \Omega_2')$ if $u_1 = u|_{\Omega_1'}$, $u_2 = u|_{\Omega_2'}$ holds. We define for $u = (u_1,u_2) \in H^1(\Omega_1' \cup \Omega_2')$

$$I_\lambda(u_1,u_2) = \inf_{w\in H^1(\mathbf{R}^N),\, w=(u_1,u_2) \text{ on } \Omega_1'\cup\Omega_2'} \Phi_\lambda(w), \tag{2.3}$$

Now we set

$$\Sigma_{i,\lambda} = \{v \in H^1(\Omega_i'); \ \|v\|_{\lambda,\Omega_i'} = 1\} \quad \text{for } i = 1,2$$

and define

$$J_\lambda(v_1, v_2) = \sup_{s,t>0} I_\lambda(sv_1, tv_2) : \Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda} \to \mathbf{R}.$$

We can observe that for any $M > 0$ there exists $\lambda(M) \geq 1$ such that for any $\lambda \geq \lambda(M)$

- For any $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda}}$, $(s,t) \mapsto I_\lambda(sv_1, tv_2)$ has a unique maximizer. This maximizer satisfies $s, t \leq \delta_M$ for some $\delta_M > 0$. Therefore $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda}}$ implies $||v_i||^{p+1}_{L^{p+1}(\Omega_i')} > \delta_M^{-(p-1)}$ $(i = 1,2)$.
- $[J < M]_{\Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda}} \to \mathbf{R}: \ (v_1, v_2) \mapsto J_\lambda(v_1, v_2)$ is of class C^1 and its critical points are corresponding to critical points of $I_\lambda(u)$.

Here we use notation:

$$[J_\lambda < M]_{\Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda}} = \{(v_1, v_2) \in \Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda}; \ J_\lambda(v_1, v_2) < M\}.$$

(b) Comparison functionals

To find critical points of $J_\lambda(v_1, v_2) : \Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda} \to \mathbf{R}$ the following observation is useful. We use notation:

$$J_{i,\lambda}(v_i) = \sup_{s>0} I_\lambda(sv_i) : \Sigma_{i,\lambda} \to \mathbf{R}.$$

Lemma 2.2. There exists $c_\lambda > 0$ such that

$$\begin{aligned}
&c_\lambda \to 0 \qquad \text{as } \lambda \to \infty, \\
&|J_\lambda(v_1, v_2) - J_{1,\lambda}(v_1) - J_{2,\lambda}(v_2)| < c_\lambda, \\
&\left|J_\lambda'(v_1, v_2)(h_1, h_2) - J_{1,\lambda}'(v_1)h_1 - J_{2,\lambda}'(v_2)h_2\right| < c_\lambda(\|h_1\|_{\lambda,\Omega_1'} + \|h_2\|_{\lambda,\Omega_2'})
\end{aligned}$$

for all $(v_1, v_2) \in [J_\lambda < M]_{\Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda}}$ and $(h_1, h_2) \in T_{v_1}\Sigma_{1,\lambda} \oplus T_{v_1}\Sigma_{1,\lambda}$. ∎

We remark that

$$\Sigma_{i,\lambda} \to \mathbf{R}: \ v_i \mapsto J_{i,\lambda}(v_i)$$

are even functionals and the existence of infinite many critical points can be obtained through minimax arguments. By Lemma 2.2, we regards $J_\lambda(v_1, v_2)$ as a perturbation of $J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2)$.

3. Proof of Theorem 1.1

In this section we give proof of Theorem 1.1. Since we bring a p close to $\frac{N+2}{N-2}$, a critical problem for $p = \frac{N+2}{N-2}$ plays an important role:

$$\begin{aligned}
-\Delta u &= u^{\frac{N+2}{N-2}} \quad \text{in } \mathbf{R}^N, \\
u &> 0 \qquad \text{in } \mathbf{R}^N, \\
u &\in H^1(\mathbf{R}^N).
\end{aligned} \tag{3.1}$$

In fact, the solution of (3.1) has a invariance under translations and dilations. Although this invariance is lost for $p < \frac{N+2}{N-2}$, the solution of (3.1) played an important role in the arguments theorem in Benci and Cerami [**BC**], Bartsch and Wang [**BW2**]

Since the index p have a important role, in this section we write dependence of J_λ, $J_{i,D}$ on p explicitly and are notation:

$$J_\lambda(p; v_1, v_2) = J_\lambda(v_1, v_2) \quad \text{for } (v_1, v_2) \in \Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+},$$

$$J_{i,D}(p; v_i) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{||v_i||_{L^{p+1}(\Omega_i)}}\right)^{\frac{2(p+1)}{p-1}} \quad \text{for } v_i \in \Sigma_{i,D,+}.$$

$$\Sigma_{i,D,+} = \{v \in H_0^1(\Omega_i);\ ||v||_{H^1(\Omega_i)} = 1, v^+ \not\equiv 0\} \quad \text{for } i = 1, 2.$$

We define

$$c_{\lambda,p} := \inf_{(v_1,v_2)\in\Sigma_{1,\lambda,+}\oplus\Sigma_{2,\lambda,+}} J_\lambda(p; v_1, v_2)$$

and

$$c_p(\Omega_i) := \inf_{v_i\in\Sigma_{i,D,+}} J_{i,D}(p; v_i).$$

By (PS)-conditions, $c_{\lambda,p}$ and $c_p(\Omega_i)$ are critical values of $J_\lambda(p; v_1, v_2)$ and $J_{i,D}(p; v_i)$ respectively.

First of all, we fix p and show two following lemmas.

Lemma 3.1. *(i) Suppose that $(v_1, v_2) \in \Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}$ is critical point of J_λ, Then corresponding critical point of Φ_λ is positive in $\mathbf{R}^N$.*
(ii) $c_{\lambda,p} < c_p(\Omega_1) + c_p(\Omega_2)$.

Proof. (i) Let $(v_1, v_2) \in \Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}$ be critical point of J_λ. Then there exists a unique maximizer $s_0, t_0 > 0$ satisfying

$$I_\lambda(s_0 v_1, t_0 v_2) = \sup_{s,t>0} I_\lambda(s v_1, t v_2).$$

We can easily show $u = (s_0 v_1, t_0 v_2)$ is critical points of I_λ. For this u, $w \in H^1(\mathbf{R}^N)$ achieving (2.3) is a solution of

$$-\Delta w + (\lambda^2 a(x) + 1)w = g(x, w) \text{ in } \mathbf{R}^N .$$

By definition of g in section 1, $g(x,u) \geq 0$. From the maximum principle it follows that $w > 0$ in $\mathbf{R}^N$.

(ii) First, since $\Sigma_{1,D,+} \oplus \Sigma_{2,D,+} \subset \Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}$, we have

$$\begin{aligned} c_{\lambda,p} &= \inf_{(v_1,v_2)\in\Sigma_{1,\lambda,+}\oplus\Sigma_{2,\lambda,+}} J_\lambda(p; v_1, v_2) \\ &\leq \inf_{(v_1,v_2)\in\Sigma_{1,D,+}\oplus\Sigma_{2,D,+}} J_\lambda(p; v_1, v_2) \\ &= \inf_{(v_1,v_2)\in\Sigma_{1,D,+}\oplus\Sigma_{2,D,+}} (J_{1,D}(p; v_1) + J_{2,D}(p; v_2)) \\ &= c_p(\Omega_1) + c_p(\Omega_2). \end{aligned}$$

Next, we show that the inequality $c_{\lambda,p} < c_p(\Omega_1) + c_p(\Omega_2)$ is strict. Suppose $c_{\lambda,p} = c_p(\Omega_1) + c_p(\Omega_2)$ and let u_i be a least energy solution of

$$\begin{aligned} -\Delta u + u &= u^p \quad \text{in } \Omega_i, \\ u &> 0 \quad \text{in } \Omega_i, \\ u &= 0 \quad \text{in } \partial\Omega_i. \end{aligned}$$

Here we set $v_i = u_i/||u_i||_{H^1(\Omega_i)} \in \Sigma_{i,D,+}$. Then $c_p(\Omega_i)$ is achieved by $v_i \in \Sigma_{i,D,+}$ and we get

$$J_\lambda(p; v_1, v_2) = J_{1,D}(p; v_1) + J_{2,D}(p; v_2) = c_p(\Omega_1) + c_p(\Omega_2) = c_{\lambda,p}.$$

Therefore $(v_1, v_2) \in \Sigma_{1,D,+} \oplus \Sigma_{2,D,+}$ achieve $c_{\lambda,p}$. But, by previous results (i), $c_{\lambda,p}$ is never achieved by for any $(v_1, v_2) \in \Sigma_{1,D,+} \oplus \Sigma_{2,D,+}$. This is contradiction. ∎

Lemma 3.2.

$$c_{\lambda,p} \longrightarrow c_p(\Omega_1) + c_p(\Omega_2) \quad \text{as } \lambda \longrightarrow \infty.$$

Proof. By previous lemma, the inequality $c_{\lambda,p} < c_p(\Omega_1) + c_p(\Omega_2)$ is strict. Let $(v_{1,\lambda}, v_{2,\lambda}) \in \Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}$ be a critical point of J_λ satisfying $J_\lambda(p; v_{1,\lambda}, v_{2,\lambda}) = c_{\lambda,p}$. Then, by Lemma 2.2 for J_λ, there exists a sequence $\lambda_n \to \infty$ and critical points $0 \not\equiv v_i \in \Sigma_{i,D,+}$ of $J_{i,D}$ $(i = 1, 2)$ such that

$$(v_{1,\lambda_n}, v_{2,\lambda_n}) \longrightarrow (v_1, v_2) \quad \text{strongly in } H^1(\Omega_1') \oplus H^1(\Omega_2').$$

and

$$J_{\lambda_n}(p; v_{1,\lambda_n}, v_{2,\lambda_n}) \longrightarrow J_1(p; v_1) + J_2(p; v_2) \geq c_p(\Omega_1) + c_p(\Omega_2)$$

Therefore,

$$c_{\lambda_n,p} \longrightarrow c_p(\Omega_1) + c_p(\Omega_2)$$

This holds without extracting subsequence. ∎

Next, in order to bring a p close to $\frac{N+2}{N-2}$, we need following lemmas. Similar lemmas showed in Benci and Cerami [**BC**].

Lemma 3.3. *For any bounded domain $\mathcal{D} \subset \mathbf{R}^N$ and $1 \le p \le q \le \frac{N+2}{N-2}$,*

$$\left[|\mathcal{D}|^{-1}\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1} c_p(\mathcal{D})\right]^{\frac{p-1}{p+1}} \ge \left[|\mathcal{D}|^{-1}\left(\frac{1}{2}-\frac{1}{q+1}\right)^{-1} c_q(\mathcal{D})\right]^{\frac{q-1}{q+1}}.$$

Where we define

$$c_p(\mathcal{D}) := \inf_{u\in H_0^1(\mathcal{D}),||u||_{H^1(\mathcal{D})}=1} \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{1}{||u||_{L^{p+1}(\mathcal{D})}}\right)^{\frac{2(p+1)}{p-1}}.$$

Proof. By using Hölder's inequality, for every $p, q \in [1, \frac{N+2}{N-2}]$ with $p \le q$ and for every $u \in H^1(\mathcal{D})$ we get

$$\int_{\mathcal{D}} |u|^{p+1} dx \le \left[\int_{\mathcal{D}} (|u|^{p+1})^{\frac{q+1}{p+1}}\right]^{\frac{p+1}{q+1}} \left(\int_{\mathcal{D}} dx\right)^{\frac{q-p}{q+1}}.$$

Hence

$$||u||_{L^{p+1}(\mathcal{D})} \le |\mathcal{D}|^{-2\frac{q-p}{(p+1)(q+1)}} ||u||_{L^{q+1}(\mathcal{D})},$$

from which we obtain

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{p+1}\right)||u||_{L^{p+1}(\mathcal{D})}^{-2\frac{p+1}{p-1}} &\ge |\mathcal{D}|^{-2\frac{q-p}{(p-1)(q+1)}}\left(\frac{1}{2}-\frac{1}{p+1}\right)||u||_{L^{q+1}(\mathcal{D})}^{-2\frac{p+1}{p-1}} \\ &= |\mathcal{D}|^{1-\frac{p+1}{p-1}\frac{q-1}{q+1}}\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{1}{2}-\frac{1}{q+1}\right)^{-\frac{p+1}{p-1}\frac{q-1}{q+1}} \\ &\quad \times \left[\left(\frac{1}{2}-\frac{1}{q+1}\right)||u||_{L^{q+1}(\mathcal{D})}^{-2\frac{q+1}{q-1}}\right]^{\frac{p+1}{p-1}\frac{q-1}{q+1}}.\end{aligned} \tag{3.2}$$

Here from definition of $c_p(\mathcal{D})$ we have

$$c_p(\mathcal{D}) \ge |\mathcal{D}|^{1-\frac{p+1}{p-1}\frac{q-1}{q+1}}\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{1}{2}-\frac{1}{q+1}\right)^{-\frac{p+1}{p-1}\frac{q-1}{q+1}} c_q(\mathcal{D})^{\frac{p+1}{p-1}\frac{q-1}{q+1}}.$$

∎

Note that $c_{\frac{N+2}{N-2}}(\mathcal{D})$ does not depends on $\mathcal{D}$, so we write $c_{\frac{N+2}{N-2}} = c_{\frac{N+2}{N-2}}(\mathcal{D})$. Moreover, $c_{\frac{N+2}{N-2}}$ is never achieved in any proper subset of $\mathbf{R}^N$.

Lemma 3.4. *For any bounded domain* $\mathcal{D} \subset \mathbf{R}^N$,

$$\lim_{p \to \frac{N+2}{N-2}-0} c_p(\mathcal{D}) = c_{\frac{N+2}{N-2}}$$

Proof. We set

$$m = \liminf_{p \to \frac{N+2}{N-2}-0} c_p(\mathcal{D}), \quad M = \limsup_{p \to \frac{N+2}{N-2}-0} c_p(\mathcal{D}).$$

By Lemma 3.3 it easily follows that

$$c_{\frac{N+2}{N-2}} \le m \le M.$$

In order to prove Lemma 3.4 we have to show that

$$c_{\frac{N+2}{N-2}} = M.$$

For any $\epsilon > 0$, by definition of $c_{\frac{N+2}{N-2}}$, we can choose a $\overline{u} \in H_0^1(\mathcal{D})$ such that

$$\frac{1}{N} ||\overline{u}||^{-N}_{L^{\frac{2N}{N-2}}(\mathcal{D})} \le c_{\frac{N+2}{N-2}} + \epsilon.$$

Next, by continuity of the map $p \mapsto ||\overline{u}||_{L^{p+1}(\mathcal{D})}$, we can choose a $\overline{p} \in (1, \frac{N+2}{N-2})$ such that for every $p \in [\overline{p}, \frac{N+2}{N-2})$,

$$\left| \frac{1}{N} ||\overline{u}||^{-N}_{L^{\frac{2N}{N-2}}(\mathcal{D})} - \left(\frac{1}{2} - \frac{1}{p+1}\right) ||\overline{u}||^{-2\frac{p+1}{p-1}}_{L^{p+1}(\mathcal{D})} \right| \le \epsilon.$$

Hence for every $p \in [\overline{p}, \frac{N+2}{N-2})$ we get

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) ||\overline{u}||^{-2\frac{p+1}{p-1}}_{L^{p+1}(\mathcal{D})} \le c_{\frac{N+2}{N-2}} + 2\epsilon.$$

This implies

$$c_p(\mathcal{D}) \le c_{\frac{N+2}{N-2}} + 2\epsilon.$$

Consequently we find $c_{\frac{N+2}{N-2}} = M$

We fix $r > 0$ such that the inclusions $\Omega_i^- \hookrightarrow \Omega_i \hookrightarrow \Omega_i^+$ are homotopy equivalences. Here we define

$$\Omega_i^+ = \{x \in \mathbf{R}^N; dist(x, \Omega_i) < r\},$$

and

$$\Omega_i^- = \{x \in \Omega_i; dist(x, \partial\Omega_i) > r\}.$$

For $v_i \in \Sigma_{i,\lambda}$, we define the center of mass of v_i:

$$\beta_i(p; v_i) := \frac{\int_{\Omega_i} |v_i|^{p+1} x dx}{\int_{\Omega_i} |v_i|^{p+1} dx}.$$

We remark that for any $\delta > 0$

$$\beta_i(p;\ \cdot\ ) : \{u \in L^{p+1}(\Omega_i'); ||u||_{L^{p+1}(\Omega_i')} \ge \delta\} \to \mathbf{R}^N$$

is continuous.

Lemma 3.5. *Assume sequences* $(p_n)_{n=1}^{\infty}$ *and* $(v_{i,n})_{n=1}^{\infty} \subset \Sigma_{i,D,+}$ *satisfy*

$$p_n \longrightarrow \frac{N+2}{N-2},$$

$$J_{i,D}(p_n; v_{i,n}) = \left(\frac{1}{2} - \frac{1}{p_n+1}\right) ||v_{i,n}||_{L^{p_n+1}(\Omega_i)}^{-\frac{2(p_n+1)}{p_n-1}} \longrightarrow c_{\frac{N+2}{N-2}}.$$

Then $\beta_i(p_n; v_{i,n}) \in \Omega_i^+$ *for large* n.

Proof. Using inequality (3.2), it follows that

$$c_{\frac{N+2}{N-2}} \leq J_{i,D}(\tfrac{N+2}{N-2}; v_{i,n})$$

$$\leq |D|^{1-\frac{p_n-1}{p_n+1}\frac{N}{2}} \frac{1}{N}\left(\frac{1}{2} - \frac{1}{p_n+1}\right)^{-\frac{p_n-1}{p_n+1}\frac{N}{2}} \left[J_{i,D}(p_n; v_{i,n})\right]^{\frac{p_n-1}{p_n+1}\frac{N}{2}},$$

from which we have

$$J_{i,D}(\tfrac{N+2}{N-2}; v_{i,n}) \longrightarrow c_{\frac{N+2}{N-2}}.$$

Here, by Ekeland's principle, there exists $(w_{i,n})_{n=1}^{\infty} \subset \Sigma_{i,D,+}$ satisfying

$$c_{\frac{N+2}{N-2}} \leq J_{i,D}(\tfrac{N+2}{N-2}; w_{i,n}) \leq J_{i,D}(\tfrac{N+2}{N-2}; v_{i,n}) \longrightarrow c_{\frac{N+2}{N-2}},$$

$$||J'_{i,D}(\tfrac{N+2}{N-2}; w_{i,n})||^* \longrightarrow 0,$$

$$||w_{i,n} - v_{i,n}||_{H^1(\Omega_i)} \longrightarrow 0,$$

as $n \to \infty$. Now, observe that from well-known compactness results (see Struwe [**St2**], Lions [**L**]), it follows that there exists $r_n \to 0$, $(x_n)_{n=1}^{\infty} \subset \Omega_i$ and solution of w_0 of (3.1) such that

$$r_n^{\frac{N-2}{2}} w_{i,n}(r_n(x - x_n)) \longrightarrow w_0(x) \text{ strongly in } H^1(\mathbf{R}^N).$$

Hence, we can show that

$$\beta_i(p_n; w_{i,n}) \in \Omega_i^+ \quad \text{for large } n.$$

Since $||w_{i,n} - v_{i,n}||_{H^1(\Omega_i)} \to 0$, we find

$$\beta_i(p_n; v_{i,n}) \in \Omega_i^+ \quad \text{for large } n.$$

∎

We set $B_r = \{x \in \mathbf{R}^N; |x| < r\}$. We remark that by the choice of r

$$c_{\lambda,p} < c_p(\Omega_1) + c_p(\Omega_1) < 2c_p(B_r),$$

so the level set

$$[J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)]_{\Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}}$$
$$= \{(v_1, v_2) \in \Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}; J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)\}$$

is not empty.

A following proposition is key proposition.

Proposition 3.6. *There exists $p_1 \in (1, \frac{N+2}{N-2})$ such that for any $p \in (p_1, \frac{N+2}{N-2})$, there exists $\Lambda_1(p) > 0$ such that $(\beta_1(p; v_1), \beta_2(p; v_2)) \in \Omega_1^+ \times \Omega_2^+$ for all $\lambda \geq \Lambda_1(p)$ and for all $(v_1, v_2) \in \Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}$ satisfying $J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)$.*

Proof. If the conclusion is not true then for any $q \in (1, \frac{N+2}{N-2})$ there exists $p \in (q, \frac{N+2}{N-2})$ and sequence $\lambda_n \to \infty$ and $(v_{1,n}, v_{2,n}) = (v_{1,n}(p), v_{2,n}(p)) \in \Sigma_{1,\lambda_n,+} \oplus \Sigma_{2,\lambda_n,+}$ such that

$$J_{\lambda_n}(p; v_{1,n}, v_{2,n}) \leq 2c_p(B_r) \quad \text{and} \quad (\beta_1(p; v_{1,n}), \beta_2(p; v_{2,n})) \notin \Omega_1^+ \times \Omega_2^+.$$

Clearly v_n are bounded in $H^1(\mathbf{R}^N)$ and $||v_{1,n}||_{L^{p+1}(\Omega_1')} \geq \delta$, $||v_{2,n}||_{L^{p+1}(\Omega_2')} \geq \delta$ by property of J_λ. We may assume

$$\begin{aligned} v_{i,n} &\rightharpoonup v_{i,0} \text{ weakly in } H^1(\Omega_i'), \\ v_{i,n} &\to v_{i,0} \text{ strongly in } L^{p+1}(\Omega_i'), \end{aligned} \tag{3.3}$$

and $v_{i,0}$ depends on p; $v_{i,0} = v_{i,0}(p)$. From (3.3), we find

$$\delta \leq ||v_{i,0}||_{L^{p+1}(\Omega_i')} \leq C||v_{i,0}||_{H^1(\Omega_i')}.$$

Furthermore, since we observe

$$\beta_i(p;\ \cdot\ ) : \{u \in L^{p+1}(\Omega_i'); ||u||_{L^{p+1}(\Omega_i')} \geq \delta\} \to \mathbf{R}^N$$

is continuous and $\Omega_1^+ \times \Omega_2^+$ is open, we find

$$(\beta_1(p; v_{1,0}), \beta_2(p; v_{2,0})) \notin \Omega_1^+ \times \Omega_2^+. \tag{3.4}$$

Since $||v_{i,n}||_{\lambda_n,\Omega_i'}$ is bounded, for any $\overline{\Omega_i} \subset \Omega_i'' \subset \Omega_i'$, we can show

$$||v_{i,n}||^2_{L^2(\Omega_i' \setminus \Omega_i'')} \leq \frac{1}{\lambda_n^2 \inf\limits_{x \in \Omega_i' \setminus \Omega_i''} a(x)} ||v_{i,n}||^2_{\lambda_n,\Omega_i'} \to 0.$$

Therefore we find

$$v_{i,n} \to v_{i,0} \equiv 0 \text{ strongly in } L^2(\Omega_i' \setminus \Omega_i''),$$

and this implies

$$v_{i,0} \equiv 0 \text{ in } \Omega_i' \setminus \Omega_i.$$

From weakly lower semi-continuous of norm, we get

$$1 = \lim_{n\to\infty} ||v_{i,n}||_{\lambda_n,\Omega_i'} \geq \lim_{n\to\infty} ||v_{i,n}||_{H^1(\Omega_i')} \geq ||v_{i,0}||_{H^1(\Omega_i)} > 0.$$

Therefore it follows that

$$c_p(\Omega_i) \le \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{||v_{i,0}||_{L^{p+1}(\Omega_i)}}{||v_{i,0}||_{H^1(\Omega_i)}}\right)^{-\frac{2(p+1)}{p-1}}$$
$$\le \left(\frac{1}{2} - \frac{1}{p+1}\right)||v_{i,0}||_{L^{p+1}(\Omega_i)}^{-\frac{2(p+1)}{p-1}}$$
$$= \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{p+1}\right)||v_{i,n}||_{L^{p+1}(\Omega_i')}^{-\frac{2(p+1)}{p-1}},$$

$$c_p(\Omega_1) + c_p(\Omega_2) \le \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{p+1}\right)\left[||v_{1,n}||_{L^{p+1}(\Omega_1')}^{-\frac{2(p+1)}{p-1}} + ||u_{v,n}||_{L^{p+1}(\Omega_2')}^{-\frac{2(p+1)}{p-1}}\right]$$
$$\le \lim_{n\to\infty} J_{\lambda_n}(p; v_{1,n}, v_{2,n})$$
$$\le 2c_p(B_r).$$

We consider a sequence $(q_k)_{k=1}^{\infty} \subset (1, \frac{N+2}{N-2})$ with $q_k \to \frac{N+2}{N-2}$ as $k \to \infty$. Applying a previous argument for each q_k, there exists a sequence $p_k \in (q_k, \frac{N+2}{N-2})$ satisfying

$$p_k \longrightarrow \frac{N+2}{N-2},$$

and we set

$$w_{i,k} := \frac{v_{i,0}(p_k)}{||v_{i,0}(p_k)||_{H^1(\Omega_i)}} \in \Sigma_{i,D,+}.$$

By Lemma 3.4, we remark $\lim_{p\to\frac{N+2}{N-2}-0} c_p(\Omega_j) = \lim_{p\to\frac{N+2}{N-2}-0} c_p(B_r) = c_{\frac{N+2}{N-2}}$. We have

$$\left(\frac{1}{2} - \frac{1}{p_k+1}\right)||w_{i,k}||_{L^{p_k+1}(\Omega_i)}^{-\frac{2(p_k+1)}{p_k-1}} \longrightarrow c_{\frac{N+2}{N-2}}.$$

According to Lemma 3.5, for large k, $w_{i,k}$ satisfies

$$(\beta_1(p_k; v_{1,k}), \beta_2(p_k; v_{2,k})) = (\beta_1(p_k; w_{1,k}), \beta_2(p_k; w_{2,k})) \in \Omega_1^+ \times \Omega_2^+$$

This is contradiction to (2.4). ∎

Lemma 3.7. *There exists $p_2 \in (1, \frac{N+2}{N-2})$ such that for any $p \in (p_2, \frac{N+2}{N-2})$, there exists $\Lambda_2(p) > 0$ such that for all $\lambda \ge \Lambda_2(p)$*

$$c_p(B_r) < c_{\lambda,p} < 2c_p(B_r)$$

Proof. By Lemma 3.2, the inequality $c_{\lambda,p} < 2c_p(B_r)$ is trivial. By Lemma 3.4, there exists $p_2 \in (1, \frac{N+2}{N-2})$ such that for any $p \in (p_2, \frac{N+2}{N-2})$,

$$|c_p(\Omega_i) - c_p(B_r)| < \frac{1}{4}c_{\frac{N+2}{N-2}} \quad (i = 1, 2),$$

and

$$|c_p(B_r) - c_{\frac{N+2}{N-2}}| < \frac{1}{4} c_{\frac{N+2}{N-2}}.$$

By Lemma 3.2, there exists $\Lambda_2(p) > 0$ such that for all $\lambda \geq \Lambda_2(p)$

$$|c_{\lambda,p} - c_p(\Omega_1) - c_p(\Omega_2)| < \frac{1}{4} c_{\frac{N+2}{N-2}}.$$

Then we get

$$\begin{aligned} c_{\lambda,p} &> c_p(\Omega_1) + c_p(\Omega_2) - \frac{1}{4} c_{\frac{N+2}{N-2}} \\ &> 2c_p(B_r) - \frac{3}{4} c_{\frac{N+2}{N-2}} \\ &> c_p(B_r). \end{aligned}$$

In order to prove Theorem 1.1, we need following lemma.

Lemma 3.8. *Let A,B,X be topological spaces and suppose that there exist maps $\alpha : A \hookrightarrow X$ and $\beta : X \hookrightarrow B$ such that $\beta \circ \alpha : A \to B$ is a homotopy equivalence. Then $cat(X) \geq cat(A)$.*

Proof. Suppose that $cat(X) = k$. Then there exist closed sets $X_1, \ldots X_k \subset X$ such that $X \subset X_1 \cup \ldots \cup X_k$ and each X_i are contractible in X. We set $A_i = \alpha^{-1}(X_i) \subset A$. It follows that

$$cat(A) \leq \sum_{i=1}^{k} cat(A_i).$$

We claim that, if $A_i \neq \emptyset$, A_i is contractible in A, that is, $cat(A_i) = 1$. Since X_i are contractible in X, there exist $H_i \in C([0,1] \times X_i, X)$ and $x_i \in X$ such that

$$\begin{aligned} H_i(0,x) &= x \quad \text{if } x \in X_i, \\ H_i(1,x) &= x_i \quad \text{if } x \in X_i. \end{aligned}$$

Furthermore, since $\beta \circ \alpha : A \to B$ is a homotopy equivalence, there exist continuous map $\varphi : B \to A$ and $G_i \in C([0,1] \times A, A)$ such that

$$\begin{aligned} G_i(0,a) &= a \quad \text{if } x \in X_i, \\ G_i(1,a) &= \varphi(\beta(\alpha(a))) \quad \text{if } x \in X_i. \end{aligned}$$

We define $F_i \in C([0,2] \times A_i, A)$ by

$$F_i(t,a) := \begin{cases} G(t,a) & \text{if } t \in [0,1] \text{ and } a \in A_i, \\ \varphi(\beta(H_i(t-1, \alpha(a)))) & \text{if } t \in [1,2] \text{ and } a \in A_i. \end{cases}$$

Then F_i satisfies

$$F_i(0,a) = a \quad \text{if } a \in A_i,$$
$$F_i(2,a) = \varphi(\beta(x_i)) \quad \text{if } a \in A_i.$$

Therefore, A_i is contractible in A, that is, $cat(A_i) = 1$. Consequently we get

$$cat(A) \leq k = cat(X).$$

∎

We show main theory.

Theorem 3.9. *Assume (a1)–(a2), (0.5) and $N \geq 3$. Then there exists a $p_1 \in (1, \frac{N+2}{N-2})$ and $\Lambda_1 \geq 1$ such that for $p \in (p_1, \frac{N+2}{N-2})$ and $\lambda \geq \Lambda_1$, Φ_λ has at least $cat(\Omega_1 \times \Omega_2)$ positive critical points.*

Proof. We may show that J_λ has at least $cat(\Omega_1 \times \Omega_2)$ positive critical points. Let $\widetilde{U} \in H_0^1(B_r)$ be a unique solution of

$$-\Delta u + u = u^p \quad \text{in } B_r,$$
$$u > 0 \quad \text{in } B_r,$$
$$u = 0 \quad \text{on } \partial B_r,$$

and we set

$$U_y(x) = \frac{\widetilde{U}(x-y)}{||\widetilde{U}||_{\lambda, B_r}} \in H_0^1(B_r(y)).$$

We note that

$$2c_p(B_r) = J_\lambda(p; U_y, U_z) \quad \text{for any } (y,z) \in \Omega_1^- \times \Omega_2^-,$$

and

$$(\beta_1(p; U_y), \beta_2(p; U_z)) = (y,z) \quad \text{for any } (y,z) \in \Omega_1^- \times \Omega_2^-.$$

Let p_1 and Λ_1 be constants given in Proposition 3.6. For any $p \in [p_1, \frac{N+2}{N-2})$ and $\lambda \geq \Lambda_1$, we define two maps by

$$\alpha(y,z) = (U_y, U_z) : \Omega_1^- \times \Omega_2^- \to [J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)]_{\Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}},$$
$$\beta(v_1, v_2) = (\beta_1(p; v_1), \beta_2(p; v_2))$$
$$: [J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)]_{\Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}} \to \Omega_1^+ \times \Omega_2^+.$$

By Proposition 3.6, we have these maps well defined and $\beta \circ \alpha(y,z) : \Omega_1^- \times \Omega_2^- \hookrightarrow \Omega_1^+ \times \Omega_2^+$ is a identity. Therefore, from Lemma 3.8 we find

$$cat([J_\lambda(p; v_1, v_2) \leq 2c_p(B_r)]_{\Sigma_{1,\lambda,+} \oplus \Sigma_{2,\lambda,+}}) \geq cat(\Omega_1^- \times \Omega_2^-)$$
$$= cat(\Omega_1 \times \Omega_2).$$

By Lusternik-Schnirelmann theory, we can show that, for any $p \in [p_1,, \frac{N+2}{N-2})$ and $\lambda \geq \Lambda_1$, J_λ has at least $cat(\Omega_1 \times \Omega_2)$ critical points. By Lemma 3.1, these critical points correspond to positive solutions. ∎

Finally, we can show that (P_λ) possesses at least $cat(\Omega_1 \cup \Omega_2) = cat(\Omega_1) + cat(\Omega_2)$ positive solutions by using Bartsch and Wang's argument in Bartsch and Wang [**BW2**]. Let $u \in H^1(\mathbf{R}^N)$ be critical points of Ψ_λ corresponding to Bartsch and Wang's solutions. Then these u satisfy $\Psi_\lambda(u) \leq c_p(B_r)$. On the other hand, let $v \in H^1(\mathbf{R}^N)$ be critical points of Ψ_λ corresponding to Theorem 3.9. By Lemma 3.7, these v satisfy $c_p(B_r) < \Psi_\lambda(v) \leq 2c_p(B_r)$. Consequently, we get Theorem 1.1.

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